ADVISORY PANEL ON CONSUMER PRICES – TECHNICAL

Second-hand cars: comparing the Time Dummy Hedonic and Homogeneous Product approaches

Status: final
Expected publication: TBC

Purpose

1. ONS have identified two potential sources of unit value bias within the planned implementation of alternative data sources for measuring second-hand car inflation – a negative bias stemming from a source of uncaptured age depreciation, and a positive bias stemming from a source of uncaptured technological improvements. These potential biases do not overtly distort the patterns captured within our indices.

2. We are looking for feedback on whether the evidence presented suggests a unit value bias, and if so feedback on our proposals for tackling this source of bias.

Actions

3. The panel are asked to:
    a. Advise on whether the evidence presented suggests a unit value bias in our planned implementation of second-hand cars.
    b. If bias exists, advise whether the proposed adjustments are an appropriate means for correcting the bias.
    c. Advise on whether our currently proposed methods (without refinements) are an improvement on existing methods and should be considered for adoption early, even if the bias cannot be mitigated.
    d. If the panel advises to switch approach towards the alternative approach (a Time Dummy Hedonic model), to advise on the approach to creating this model.

Background

4. In November 2022, we (the Office for National Statistics) presented to the Advisory Panels for Consumer Prices a readiness assessment for introducing alternative data sources for rail fares and second-hand cars in 2023. Using the feedback from the meeting, we made a "provisional go" decision to enter production with both datasets provided the necessary systems development could be completed by a January deadline.

5. Around the same time, we were comparing our proposed Homogeneous Product[1] (HP) approach to a Time Dummy Hedonic (TDH) alternative to investigate whether unit value bias exists within our HP approach. (Note that our HP approach uses the 25-month window GEKS-Törnqvist with a mean splice on published extension method.)

6. Initial results in December 2022 were promising and suggested both the HP and TDH approaches were very similar (as we will see in our "benchmark results"). However, further investigations uncovered potential small biases due to the treatment of age and trim variables, which we will discuss in this paper.

[1] We have sometimes referred to this as "product grouping". The "homogeneous product" term has become international nomenclature so we will refer by this term instead.

7. In January 2023, we opted to proceed with entering production with rail fares, but not second-hand cars data. The primary reason for this was due to remaining systems build and testing that was required for second-hand cars, but unnecessary for rail fares.

8. However, considering the potential evidence of unit value bias, we now need to consider how best to approach this.

What is unit value bias and how could it feature in the data?

9. When we divide the total expenditure sold for a product (or group of products) by the total quantity sold, then we calculate a **unit value**. A unit value can give a more accurate representation of the average transaction price (when compared to traditional point-in-time prices) and is often preferred when working with bigger, alternative data sources, *under the condition that the transactions are homogeneous*.

10. If transactions are homogeneous, then product quality associated with each individual transaction is approximately equal and therefore the unit value is unlikely to shift due to quality change. For example, every person buying a tin of baked beans of a specific brand and weight within a particular retailer is likely to receive a product of near-identical quality. In this instance it is unlikely for quality changes to affect the index.

11. By contrast, if the transactions are heterogeneous, then there is a risk of unit values shifting due to compositional changes with respect to quality, rather than price change. When this occurs, we describe it as unit value bias. For example, if we were to aggregate the transactions of both single tins of beans and multipacks into a unit value, then the unit value can shift due to the proportion of consumers purchasing singles and multipacks each month. These shifts will cause the index to be influenced by compositional consumption of quality.

12. The homogeneous product (HP) approach uses a matched-model index method to measure price change in the unit values of homogeneous products. This approach is straightforward in markets such as groceries where goods are mass-produced, and we can track prices over time using a product identifier.

13. However, the HP approach is less straightforward in the second-hand car market where no two cars are the same since even cars of the same model experience different levels of depreciation. Instead of using a product identifier, we group together products of similar quality based on a variety of quality characteristics. The goal is to produce a definition that is tight enough to control for quality but avoids being so tight that we are unable to form the product matches needed in a matched-model index.

14. In a previous paper, we proposed a tight product definition for second-hand cars, optimising the MARS metric to account for both homogeneity and product match rates (see "Defining a product" here). Due to the tight definition, we thought the risk of unit value bias to be small.

15. Most of the variables used in the strata/product definition are categorical and therefore unable to vary in quality. However, two variables accounting for depreciation due to usage, age and mileage, had to be introduced within "bins" to allow sufficient product matches to form indices. If fluctuations in quality occur within these bins then there is some risk of unit value bias. For example, for a product defined as a "petrol, 1-2 year old, Ford, Fiesta, Mk8, 1.4L, 20,000-30,000 mile, manual, hatchback", the unit value in January could be associated with a set of transactions with an average age of 1 year and 5 months, whereas in June it could have an average age of 1 year and 6 months. This quality reduction would therefore be unaccounted for, resulting in a unit value bias.

16. We can use the Time Dummy Hedonic (TDH) model to avoid using unit values, by using observed prices in a regression-based unmatched model. Terms like age and mileage can be introduced into the model continuously rather than through banding, which may allow us to better account for changes that may occur within these bands.

17. In Annex A, we have provided some guidance on how the TDH model is calculated.

How we will compare the Homogeneous Product and Time Dummy Hedonic approaches

18. Our analysis comparing the Homogeneous Product (HP) and Time Dummy Hedonic (TDH) approaches will cover 44 months between January 2019 and August 2022.

19. For both approaches we use the same strata – fuel type, age band and car make. There are 359 elementary aggregates due to combinations of these strata, but we remove 24 of the smaller elementary aggregates within the TDH to avoid imputation problems. These 24 aggregates only account for 0.3% of expenditure[2].

20. Since the TDH is a multilateral index method, it is paired with an extension method to avoid the need for revisions. We have discussed this previously in the context of the GEKS-Törnqvist. For this review, to be consistent with the HP/GEKS-Törnqvist approach, we use a 25-month window and a mean splice on published extension method. However, if we were to decide to switch towards the TDH for production use, we may need to re-explore our model choices within the context of the TDH approach.

21. Note that using a 25-month window over 44 months results in 20 TDH models per elementary aggregate (each model representing a separate 25-month window). Since we have 335 elementary aggregates with our strata choices, we compile a total of 6,700 TDH regression models for these analyses. This has made the manual elements of quality assuring the models impractical, and this will remain a major obstacle for use of the TDH at scale if chosen for production. Even so, our congruous HP and TDH benchmark results suggests this does not seem to be a major concern for the purpose of our analyses in this paper.

22. Given our choice of strata, the number of observations used to fit a 25-month TDH regression model can vary greatly with approximate minimums and maximums of 400 and 84,000 respectively. The median number of rows used is 11,000.

23. For both HP and TDH approaches, after computing the elementary aggregates we use the Lowe index method to aggregate to the fuel type consumption segment level (the lowest level we plan to publish on). Weights are calculated from the data as expenditure shares of each strata from the previous year. These are updated annually using chaining.

24. For the HP approach, as discussed in our research paper (see “Defining a product” here) we will construct homogeneous products within a strata using the following variables:
    - Model
    - Mark
    - Transmission
    - Engine size
    - Body type
    - Mileage bands

[2] Note that as discussed in our previous second-hand car papers, we approximate the sale price of a vehicle from its last observed listing price before the listing is withdrawn from the website. Therefore, our estimation of expenditure is an approximation.

25. We will compare our final HP indices produced in our impact analysis to various specifications of the TDH model.

26. We have identified a discontinuity in the "mark" variable within the last two months of our analyses, affecting both the HP and TDH results. This is due to a change in processing of this variable by Auto Trader. Further investigation suggests that the mark variable can be of inconsistent quality in general and in our future work we will look to address this problem through standardisation. Note that the first 42 months of this analysis are unaffected by this discontinuity, and so our conclusions are largely unaffected.

Definitions

27. For our results we will use a few bespoke terms. **Average sub-model $\boldsymbol{R^2}$** is an average of the elementary aggregate R^2 values, telling us the proportion of variance within the elementary aggregates explained by the model. We average first across the window R^2 values for each elementary aggregate and then average across the elementary aggregates.

28. **Model $\boldsymbol{R^2}$** is a single R^2 value calculated using predicted and actual price comparisons across all the TDH sub-models simultaneously. This will allow the $\boldsymbol{R^2}$ to be affected not only by the performance of the TDH models, but also by the granularity of our strata definitions. Whereas the average sub-model R^2 tells us the proportion of variance of our elementary aggregates explained by our TDH models, the Model R^2 value tells us the proportion of variance of second-hand cars explained by our stratification and TDH sub-models.

29. **Median parameter count** measures the number of parameters used in the TDH sub-models. We first calculate an average number of parameters over the 20 windows for each elementary aggregate, and then calculate a median over these elementary aggregates. Note that many of the categorical variables are introduced into the regression models as dummies, which can result in a higher-than-anticipated number of parameters.

30. We will use "+" to indicate when variables are concatenated within our TDH models. For example, "model + mark" would indicate that the model variable and the mark variable are concatenated first before being introduced into the model. (Note that the mark is often meaningless without the model.)

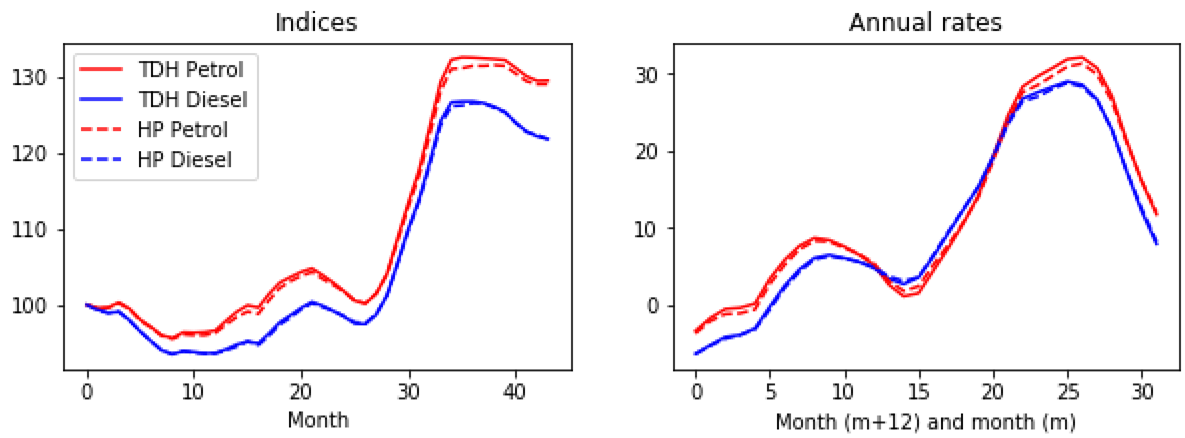
Results: mileage results and benchmark comparison

31. We start with a benchmark comparison, where the strata and quality variables used within the HP approach will also be used in our TDH:
    - Strata: fuel type; age band; make
    - Variables: model + mark; transmission; engine size; body type; mileage (continuous)
    - Average sub-model R^2: 0.869
    - Model R^2: 0.966
    - Median parameter count: 27

32. Note that the main difference in the variables used between the two approaches is that HP uses a banded mileage variable whereas TDH uses a continuous mileage variable.

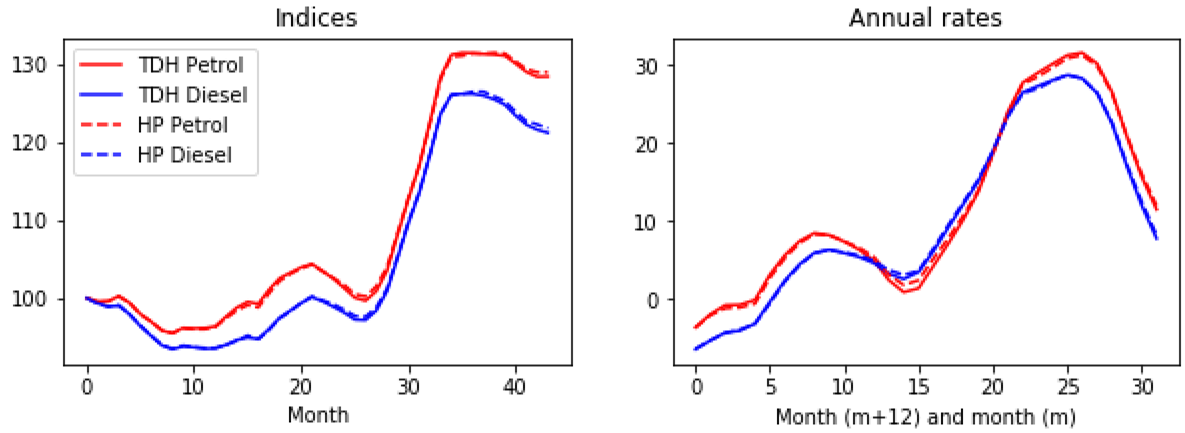
33. This benchmark comparison is shown in Figure 1. Note that differences between HP and TDH are extremely small.

Figure 1. When similar strata and quality variables are used, both the HP/GEKS-Törnqvist and TDH give very similar results



34. Within second-hand cars we are trying to capture two distinct forms of quality:
    a. Core quality – when bought new, the extent to which two models of car are different in quality due to the core characteristics of each car.
    b. Depreciation quality – the extent to which the car is of a lesser quality due to depreciation caused by usage.
35. Instead of fitting the "core quality" parameters independently, we could concatenate these variables. For example, rather than fitting separate parameters associated with whether the product is a (Ford) Fiesta and whether the product has a 1L engine, we could instead fit a single parameter associated with whether the product is a "Fiesta with a 1L engine". This may be preferable since when concatenating a lot of these variables (such as an "automatic Ford Fiesta mk8 hatchback with a 1L engine"), we can identify a specific car and account for quality variables beyond those explicitly used.
36. We therefore consider a second benchmark model where the "specification quality" variables are fully concatenated:
    - Strata: fuel type; age band; make
    - Variables: model + mark + transmission + engine size + body type; mileage (continuous)
    - Average sub-model R^2: 0.877
    - Model R^2: 0.969
    - Median parameter count: 41
37. Concatenating the core quality variables mildly improves the R^2 score and as shown in Figure 2, makes HP and TDH even more consistent, albeit at the cost of fitting additional parameters.

Figure 2. Concatenating the core quality variables makes the TDH nearly fully consistent with HP



38. Given the high degree of consistency of this second benchmark with the HP approach, we do not believe there is evidence of significant unit value bias due to the banding of mileage over the period shown.

39. In our further analyses, we will continue to concatenate the "core quality" variables. However, the results are largely unaffected by concatenation.

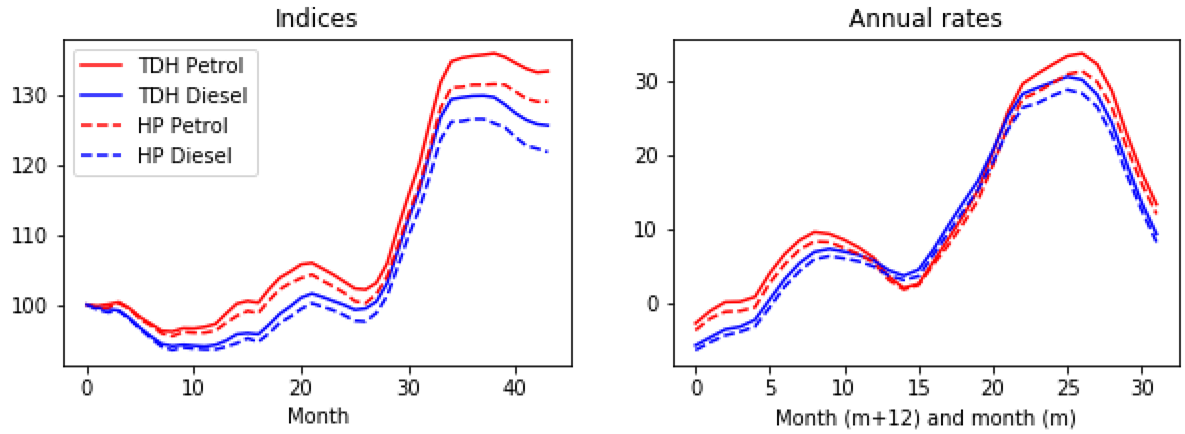
Results: Do we need an additional, more-granular age term?

40. Another potential concern is whether unit value bias is introduced by using age bands within strata selection. For example, this may occur if the aggregate age of the cars within the "Petrol, 1-2y, Skoda" strata changes over time.

41. To investigate this, we expand on the second benchmark model to introduce an additional continuous age variable:
    - Strata: fuel type; age band; make
    - Variables: model + mark + transmission + engine size + body type; mileage; age
    - Average sub-model R^2: 0.884
    - Model R^2: 0.971
    - Median parameter count: 42

42. Figure 3 shows that introducing this age term causes an upwards effect on the indices and annual rates. The median difference in annual rates between TDH and HP is 1.3 percentage points for Petrol cars and 1 percentage point for Diesel cars.

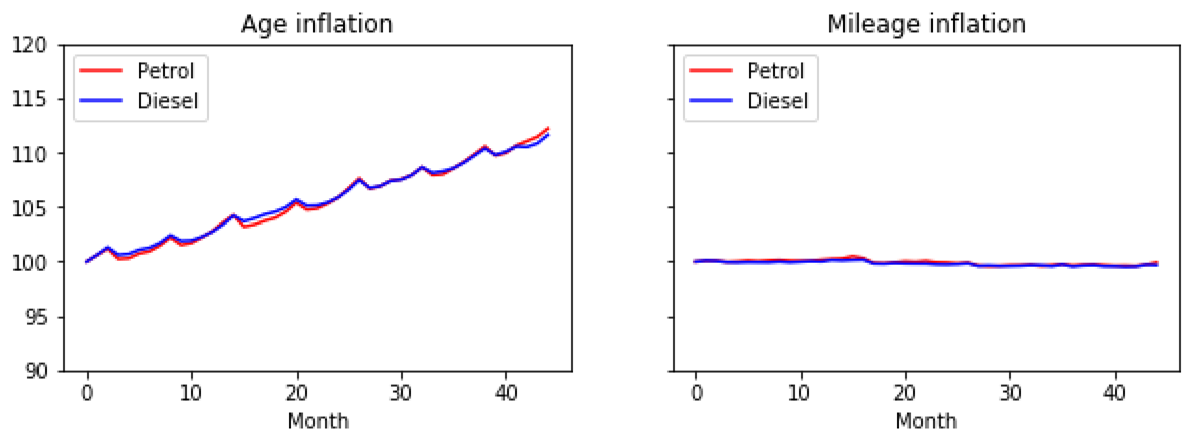
Figure 3. Introducing a more granular age term causes an upwards effect on indices



43. A potential explanation for why introducing the age term has changed the indices might be because age is proxying other quality characteristics. Age is a complex variable that not only accounts for the usage of the vehicle but may also account for generational evolution of car models. This explanation is not satisfying since we would expect a downwards effect on indices if introducing the age term was quality-adjusting for technological improvements.

44. In fact, we have evidence to suggest that despite the age-based strata, the products comprising our sample are still aging. To show this, we produced an "age index" (as well as a "mileage index"). We defined a product using the combination of strata and quality variables used in HP and measured a weighted age (and mileage) change index using a GEKS-Törnqvist. If the age and mileage bands in the product definition fully controlled for age and mileage, we might expect flat indices. Figure 4 shows this does seem to occur in the case of mileage, but not in the case of age – which may explain why use of a continuous mileage variable in the TDH does not affect the indices too much, but introducing a continuous age variable does.

Figure 4. The products defined according to the HP approach appear to be experiencing age inflation



45. We have two potential explanations for why our homogeneous products may be aging. Firstly, it may be due to market changes. As Statista shows, the number of registered new cars fell between 2016 and 2022, which may have also distorted the availability of differently-aged cars within the second-hand car market. If this explanation is true, then this effect is only likely to exist whilst the market adjusts.

46. Secondly, it may be due to the way ages interact with the strata. Consider a new car model that gets released into the new car market in January 2022. Within the second-hand car market, in January 2023, the 1-to-2-year strata can only contain 12-month-old cars for that product since anything over 12 months ago would have been before the car model's first release date. In February 2023, there will be a mix of 12- and 13-month-old cars and in March 2023 there will be a mix of 12-, 13- and 14-month-old cars (and so on). This would cause the product to "age" over time within the strata.

47. Note that there are also biannual peaks within Figure 4 occurring in March and September, corresponding with the release of new number plates. This is often a busy time in the car market when consumers often purchase new vehicles and sell their used vehicle. These peaks indicate an additional seasonal pattern in car age compositions.

48. We believe that not accounting for age on a more granular level means that some age depreciation is not being accounted for, leading to a relatively mild unit value bias.

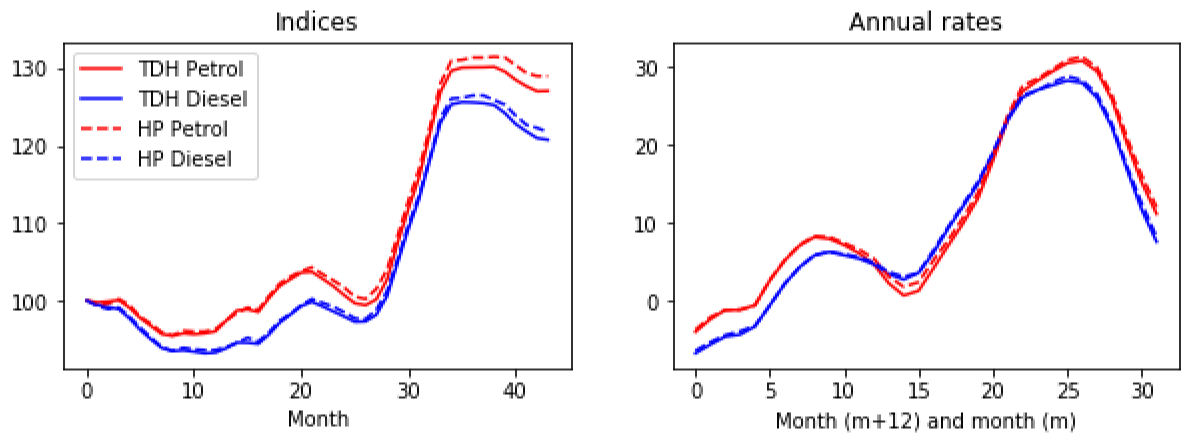
Results: Do we need to account for the trim variable?

49. Cars can typically be identified by their make, model, mark/generation and trim. Each generation of car typically has a base trim, along with additional trims offering "optional upgrades" to characteristics such as interior upholstery, supplementary technology (such as parking sensors) and exterior appearance.

50. We therefore expand on the previous model by introducing the trim variable:
    - Strata: fuel type; age band; make
    - Variables: model + mark + trim + transmission + engine size + body type; age; mileage
    - Average sub-model R^2: 0.935
    - Model R^2: 0.986
    - Median parameter count: 127

51. Introducing the trim variable causes a moderate improvement to the R^2 values. As shown in Figure 5 (when compared to Figure 3), there is a reasonably large downwards change to the indices from introducing the trim variable. This may be unsurprising since "upgrade trims" are often released later than their base variant, so adding the trim variable is likely to be adjusting for quality over time.

Figure 5. Introducing the trim variable causes the indices to fall, potentially by accounting for technological improvements made as new trims are released



52. However, note that introducing the trim has caused the median number of variables used to increase from 42 to 127! There are a lot of combinations of model, mark and trim for each car make and each of these combinations needs to be introduced as a dummy. It is possible that introducing so many terms is causing overfitting, diluting the effect on the age parameters. Note that since we are fitting with a lot of data (with a median of 11,000 rows per model), we do have some protection against overfitting.

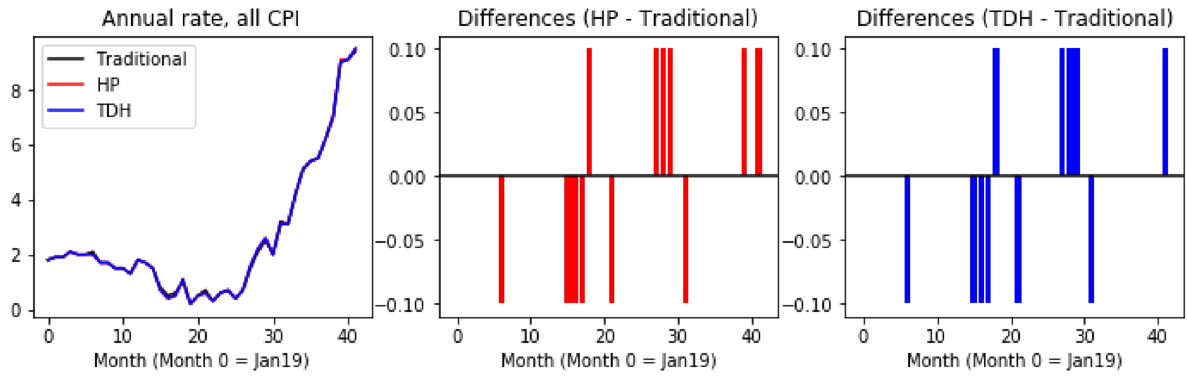
53. This is one of the key challenges with using a TDH approach. Accounting for make, model, mark and trim means either having:
    a. Many granular strata with fewer quality variables, which may increase the risk of model failure and the need for imputation due to lack of observations, or
    b. Fewer strata with many variables, carrying a higher risk of potential multicollinearity.

54. Note that the current balance of using fuel type, age band and make as the strata definition will cause us to need to fit, maintain and interpret (335*13) 4,335 regression models a year.

Impact of changes on headline CPI

55. We now consider the impact that accounting for these biases would have on our headline CPI results. In Figure 6 we compare headline CPI results from our traditional, HP and TDH approaches. As can be seen, HP (before accounting for age and trim) and TDH (after accounting for age and trim) would have made the same changes (after rounding) to the headline CPI result.

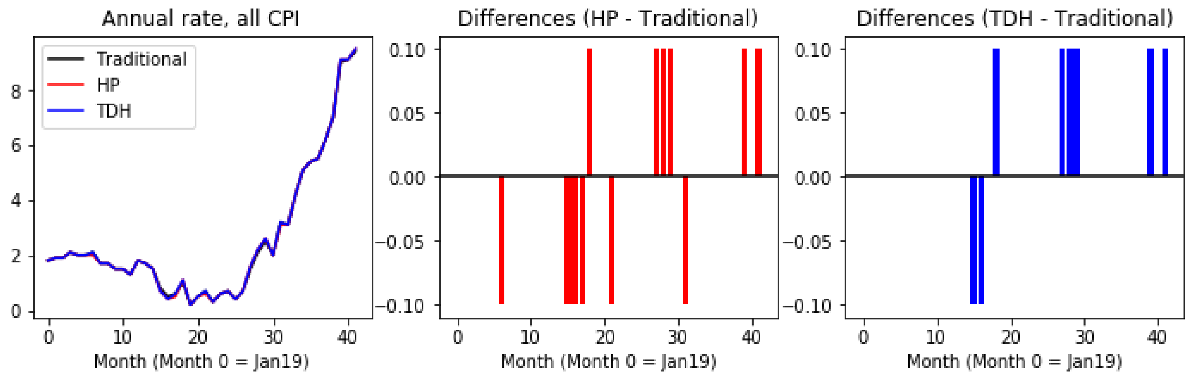
Figure 6. Headline CPI results, comparing traditional, HP and TDH, where TDH accounts for the aforementioned age and trim biases



56. We caveat our results since our HP results contain the rail fares transformations, whereas the TDH results do not. As can be shown in our previous impact analysis (see Figure 2), transformation of rail fares makes a negligible impact on the headline level, so this is not considered a concern.

57. Note that the potential age and trim biases are competing: not accounting for age results in negative bias, and not accounting for trim results in positive bias. These biases “cancel” one another out. However, this leads to a potential where one of the biases gets minimised and not the other, leading to a bigger headline impact from the other bias. In Figure 7 we therefore consider the impact on headline when accounting for the age, but not the trim bias. As can be seen, this would have only resulted in small 0.1 changes in four of the 44 months – suggesting a very mild impact even in this “worst case scenario”.

Figure 7. Headline CPI results, comparing traditional, HP and TDH, where TDH accounts only for the age bias



58. We therefore believe that whilst we should aim to minimise these biases, they do not appear to be a major problem.

Can the HP approach be refined to solve these biases?

59. In this paper we have noted two sources of potential unit value bias in the HP approach coming from banding the age variables, and not using the trim variable.

60. Accounting for age is challenging. Introducing a more-granular monthly age term into the HP product definition is not viable since match rates would drop too low.

61. However, a potential solution may be to first adjust the prices for age (and potentially mileage) prior to using the HP approach. This would allow us to account for age without introducing it directly into the product definition. One potential way of doing this could be using similar interpolation techniques used in the traditional method. We could potentially calculate an average proportion of the value of a car lost each month due to age depreciation and adjust the prices of all cars within an age band to be of a consistent age.

62. By contrast, the trim variable appears straightforward to introduce into the HP model. Our original decision not to include this variable was based on MARS analysis that showed the degradation in the match rate element of MARS outweighed the improvement in the R^2. However, avoiding this bias seems more important, so we can explore the effect of introducing the trim variable into the model.

63. We are particularly interested in the panel's feedback on whether these recommended refinements seem viable, or to otherwise advise alternative approaches.

Comparing with the traditional method

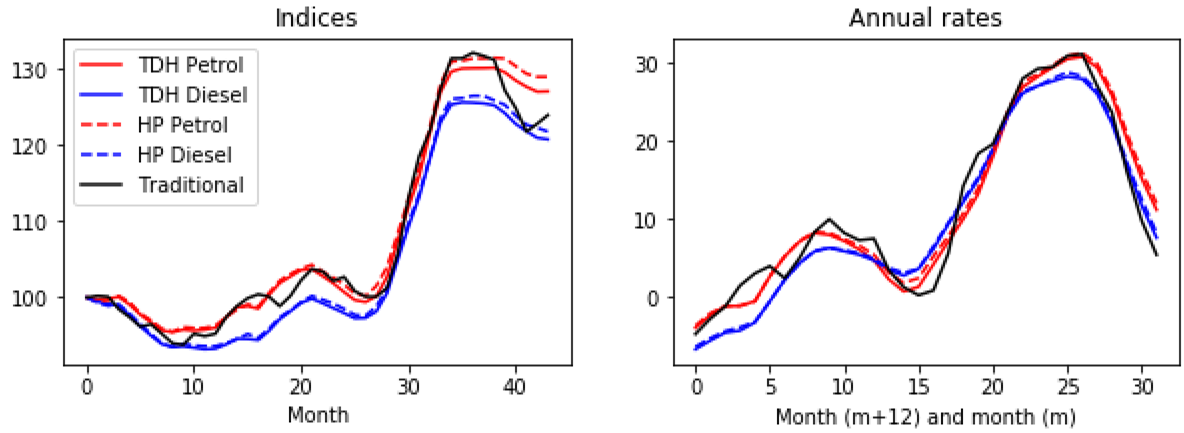
64. Despite the challenges outlined in this paper, we should consider whether implementation of alternative data sources with our currently proposed methods offers an improvement on traditional measurement of second-hand cars.

65. We can summarise[3] the traditional approach as:
    - A sample of 35 cars is selected, drawn with probabilities proportional to size.
    - "Advised selling prices" for the cars are obtained from Glass's guide, based on an average of seller asking prices.
    - Prices are collected for one-, two- and three-year old cars.
    - As the cars age throughout the year, we interpolate between the prices of the differently aged cars to obtain the price of a consistent age.
    - Prices are adjusted to account for model changes by using ratios from overlapping prices between new and old models.
    - We publish item indices for two- and three-year old cars.

66. In Figure 8 we compare our final TDH model, our HP approach, and our traditional approach. As can be seen, the differences between TDH and HP are smaller compared to the traditional series. The traditional approach does track the general trends observed in the other series but exhibits greater volatility.

Figure 8. We compare our TDH, HP and traditional approaches to measuring second-hand car inflation

[3] A more detailed summary can be found in section 9.5.3.1 of our Consumer Prices Technical Manual.



67. There are several reasons to prefer our proposed HP approach to traditional methods. The improvement in sample sizes reduces volatility and the improvement in granularity allows us to publish indices broken down by fuel type. We can use cars ranging in age from one to ten years, rather than only two- and three-year old cars.

68. Therefore, we consider implementing HP with or without the refinements is likely to be an improvement on existing methods.

Next steps

69. Following the Advisory Panels, we will meet internally to decide the short-term (by 2024) and long-term (beyond 2024) plans for implementing second-hand cars.

70. In the short-term, we will consider the following options:
    a. Introduce our HP approach without refinements (as a transitional improvement).
    b. Delay implementation of second-hand cars until these biases can be addressed.

71. In the long-term, we will consider the following options:
    a. Use HP with the refinements discussed previously.
    b. Use TDH with the final model specification outlined in this paper.
    c. Use TDH with a new model specification (this would require further research).

72. Introducing the refinements to the HP solution may be viable by 2025. However, switching to the TDH is likely to cause a longer delay beyond 2025 as this would require substantial further research and development, and our milestones are currently prioritising the implementation of grocery scanner data.

73. If we can refine the HP approach to better adjust for age (and potentially mileage) changes, we would favour the continued use of the HP approach. We prefer HP for several reasons:
    - HP does not require substantial regression model maintenance.
    - HP is easier to interpret, carrying less concern that indices aren’t being affected by overfit models.
    - HP uses the GEKS-Törnqvist, consistent with the implementation of other alternative data sources.
    - HP is a more accessible method which is easier to communicate.
    - Switching to the TDH will require substantial pipeline development resource.

74. We are interested in the panel’s feedback on these plans.

Further work

75. In our explorations of the TDH, we have noticed that the mark/generation variable can be of inconsistent quality. Sometimes multiple descriptions exist for the same mark (especially as the variable often contains codes such as “B299” and descriptions such as “Mark 6” which are equivalent). This is likely degrading the match rate in the HP approach. In our TDH analyses making some manual improvements to this variable improved the predictive performance of the model. Therefore, we will explore whether we can improve this variable further, whether by manual means, automated mappings or machine learning.

76. Given a car make, model and mark, we obtain the following uniformity statistics (the extent to which a variable is of a single value) for the following variables:
    a. Transmission: 78%
    b. Fuel type: 76%
    c. Emissions: 90%
    d. Drivetrain: 90%
    e. Body type: 95%
    f. Seats: 99%
    g. Engine power: 86%
    h. Engine size: 83%

77. In future work we can use these uniformity statistics to advise on variables that have already been largely accounted for within the strata/product definition. In the TDH this will help us avoid overfitting from introducing too many variables, and within the HP approach this will help us avoid loss of matches by avoiding introducing highly uniform variables.

78. If we choose to continue work on the TDH then we may also investigate transformations of the mileage and age variables further.

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Prices Division and Methodology
April 2023

List of Annexes

Annex A: How is the Time Dummy Hedonic model calculated?

79. The Time Dummy Hedonic (TDH) method involves predicting log prices from a regression model that accounts for both time (covering a window of months from month 1 to month T) and quality characteristics:

$$\ln\left(p_i^t\right) = \alpha + \sum_{t=2}^{T} \delta^t D_i^t + \sum_{k=1}^{K} \beta^k v_i^k + \varepsilon_i$$

Where:

D_i^t are time dummy variables for months t={2,…,T}

v_i^k are a set of k={1,…,K} quality variables

$\alpha, \delta^t, \beta^k$ are model coefficients

ε_i is an error term

80. We estimate the coefficients using WLS (using expenditure shares for weights), obtaining:

$$\ln(\hat{p}^t) = \hat{\alpha} + \sum_{t=2}^{T} \hat{\delta}^t D^t + \sum_{k=1}^{K} \hat{\beta}^k v^k$$

81. Consider month $t = m > 1$. In this case, $D^{t=m} = 1$ and $D^{t \neq m} = 0$. We can therefore reduce the previous equation to:

$$\ln(\hat{p}^{t=m}) = \hat{\alpha} + \hat{\delta}^{t=m} + \sum_{k=1}^{K} \hat{\beta}^k v^k$$

82. Similarly for month $t = 1$, all dummies are zero, and therefore we get:

$$\ln(\hat{p}^{t=1}) = \hat{\alpha} + \sum_{k=1}^{K} \hat{\beta}^k v^k$$

83. Exponentiating both sides of the previous two equations give:

$$\hat{p}^{t=m} = \exp\left(\hat{\alpha} + \hat{\delta}^{t=m} + \sum_{k=1}^{K} \hat{\beta}^k v^k\right) = \exp(\hat{\alpha}) \exp\left(\hat{\delta}^{t=m}\right) \prod_{k=1}^{K} \exp\left(\hat{\beta}^k v^k\right)$$

$$\hat{p}^{t=1} = \exp\left(\hat{\alpha} + \sum_{k=1}^{K} \hat{\beta}^k v^k\right) = \exp(\hat{\alpha}) \prod_{k=1}^{K} \exp\left(\hat{\beta}^k v^k\right)$$

84. Finally, we calculate the TDH for month m, with respect to base month 1, defined within a window covering months 1 to T, as:

$$P_{TDH}^{(1,m)} = \frac{\widehat{p}^{t=m}}{\widehat{p}^{t=1}} = \frac{\exp(\widehat{\alpha}) \exp\left(\widehat{\delta}^{t=m}\right) \prod_{k=1}^{K} \exp\left(\widehat{\beta}^k v^k\right)}{\exp(\widehat{\alpha}) \prod_{k=1}^{K} \exp\left(\widehat{\beta}^k v^k\right)} = \exp\left(\widehat{\delta}^{t=m}\right)$$

$$P_{TDH}^{(1,1)} = \frac{\widehat{p}^{t=1}}{\widehat{p}^{t=1}} = \frac{\exp(\widehat{\alpha}) \prod_{k=1}^{K} \exp\left(\widehat{\beta}^k v^k\right)}{\exp(\widehat{\alpha}) \prod_{k=1}^{K} \exp\left(\widehat{\beta}^k v^k\right)} = 1$$

85. As a multilateral index, the window of indices is therefore defined as:

$$P_{TDH}^{(1,1)}, P_{TDH}^{(1,2)}, P_{TDH}^{(1,3)}, \ldots, P_{TDH}^{(1,T)} = 1, \exp\left(\widehat{\delta}^2\right), \exp\left(\widehat{\delta}^3\right), \ldots, \exp\left(\widehat{\delta}^T\right)$$